HS 206r, Spring 2012, Connemara Doran

Pappus (4th century AD) on the Quadratrix (*Collection* Book IV, Propositions 26-29)

The **quadratrix** is a motion-generated curve used as a mathematical "tool" to extend what is constructible beyond what is possible with straight-edge and compass alone. As originally conceived (as early as 5th century BC), the quadratrix was the locus of points generated by the moving point of intersection of two line segments in synchronized motion – namely, a vertical side of a square pivoting about a lower vertex to trace a quadrant of a circle, and concurrently the top side of the square remaining horizontal while falling at a proportional rate, so that the two intersecting lines coincide in their arrival along the base of the square. If one accepts the legitimacy of the quadratrix as a constructive tool, it makes possible

- trisecting the angle – constructing an angle one-third of a given angle (in fact, constructing any number of equal divisions of an angle)
- doubling the cube – constructing the edge of a cube with volume double that of a given cube
- squaring the circle – constructing the side of a square with area equal to that of a given circle

The name quadratrix means "squarer" and was so named because its principle "symptom" is the generation – as the end product of the motions – of a unique point and hence "line segment" by means of which one can construct the **square equal to the circle** (of radius equal to the side of the original square).

According to Heike Sefrin-Weis (250), Propositions 26-29 on the quadratrix by Pappus of Alexandria are "of the **highest importance for the history of ancient mathematics**," for

- Propositions 26 and 27 are the **"only surviving evidence [of] this curve (and the squaring of the circle with it) from antiquity,"** and
- Propositions 28 and 29 "are our only extant detailed examples for an **analysis of loci on surfaces** (used here for the **geometrical justification/description of the *genesis* of the quadratrix**)"

However, the generation of the quadratrix by these synchronized motions (used for Propositions 26 and 27) was **highly contested** because of the **problematic status** of the generated point of the quadratrix at the end of the process – namely, when the two intersecting lines arrive in sync at the bottom, they no longer intersect (they become parallel). So what was problematic was the existence of the "limit" of the process generating the quadratrix and hence the validity of the measure of the unique "line segment" with which the circle is squared. The critic Sporus (3rd century AD) also argued that the generation of the quadratrix by these motions assumes what it is trying to prove (as explained below).

As developed further in the 4th and 3rd centuries BC, Propositions 28 and 29 **seek to resolve this issue of the generation of the quadratrix** by a "geometrical analysis" which generates the quadratrix inside a "configuration" of surfaces and curves created by controlled motions (260-265). Key to this analysis were a series of decisions regarding the **"givens"** – working in three-dimensional space (a quadrant of a right cylinder [28] or a cylindroid over spiral [29]); assuming as given a well-understood mathematical curve (a helix [28] or Archimedean spiral [29]); and taking as given **an arbitrary ratio** [specifically not taking as given the ratio of the circumference to the radius of a circle]. A surface is created, the projection of which on the base of the quadrant of the cylinder traces a curve; in the **particular case** where the ratio is that of the circumference to the radius, this uniquely determined curve is the quadratrix.

Our understanding of the context of Pappus' text, and of the history of the ancient development of the quadratrix, is limited. Pappus (4th century AD) states that Dinostratus (4th century BC) and Nicomedes (3rd century BC) used the quadratrix to square the circle. Although Heath attributes the quadratrix to Hippias of Elis in the fifth century BC, Knorr argues that the mathematical sophistication required would not have been available at that time, a claim which Sefrin-Weis considers "a bit too pessimistic" since Hippias could have invented the curve itself (and possibly its use in angle trisection) without recognizing the rectification property (Sefrin-Weis, 248-9). The proof which Pappus gives for Proposition 26 is "almost certainly post-Archimedean" (3^{rd} century BC or later) since it implicitly uses Archimedes Proposition 1 from *Measurement of the Circle* (248). But Sefrin-Weis thinks it is unlikely that Archimedes was the source for Proposition 29, since it does not appear to be "Archimedean" in its analytic approach, which is distinct in style from the "quasi-mechanical methods, and perhaps infinitesimals" favored by Archimedes (249).

Generation (*Genesis*) of the quadratrix via "synchronized motions":

Let ABCD be a square and describe the arc BED of a circle with center A.

Let AB (side of the square and radius of the circle) **rotate uniformly** about A, with A fixed and with B moving along the arc BED.

Simultaneously (synchronously with the motion of side AB), let side BC **descend uniformly**, always remaining parallel to AD (i.e., remaining horizontal).

In the space between sides AB and AD and arc BED, **the intersection** of the synchronously moving lines AB and BC **traces out a curve** [the **quadratrix**] BZH which is concave in the same direction as BED and which "appears to be useful for finding a square equal to the given circle."

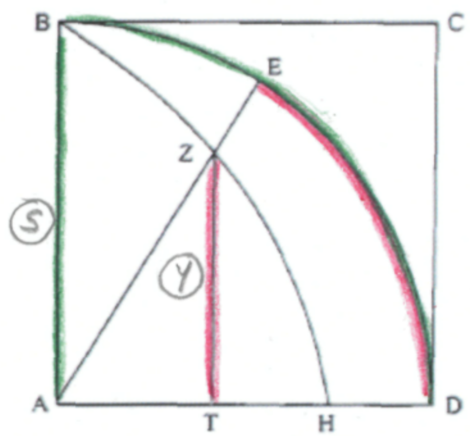


Figure 1: curve BZH is the quadratrix (from Sefrin-Weis)

Principal *Symptoma* (characteristics, relationships embedded in the curve) of the quadratrix:

The principle property (*symptoma*) of the quadratrix is this:
If any straight line, such as AZE, is drawn from A to the arc BED, the ratio of the whole arc BED to the arc ED will be equal to the ratio of the side AB to the line ZT.

That is, letting s be the given length of a side of the square (AB), and y be the length or "height" ZT which varies during the movement,

$\frac{arc\ BED}{arc\ ED} = \frac{AB}{ZT}$ i.e., $\frac{arc\ BED}{arc\ ED} = \frac{s}{y}$

[arc BED is one-fourth of the circumference of the circle (2πs), and so is of length $\frac{\pi}{2}s$]

Sporus' Criticism of the *Genesis* of the Quadratrix

The goal for which the construction seems to be useful is assumed in the hypothesis.

In order for the two lines to be in synchronized motion, one must know the ratio of the straight line AB to the arc BED in advance [i.e., one must know the value of π, "or else use an approximation to stand in for it," Sefrin-Weis, 132, footnote 4]. And how could the motions end together (halt simultaneously), using unadjusted (indeterminate) speeds, unless it might happen sometime by chance?

The endpoint which is used for the squaring of the circle – i.e., the point at which the quadratrix "intersects" the base AD (i.e., point H) – is not found [by the above generation]. For when AB and BC stop simultaneously, they will coincide with AD and will no longer intersect each other.

[Therefore,] the point H can only be found [known with certainty] by assuming the ratio of the circumference to the radius [i.e., knowing the value of π and constructing a straight line proportional to that length].

"So **unless this ratio is given**, we must beware lest, in following the authority of those men who discovered the line, we admit its construction, which is **more a matter of mechanics**" (Loeb translation, 341). Alternatively, this could be translated as: "**Without this ratio being given**, however, one must not, trusting in the opinion of the men who invented the line, accept it [as fully geometrical], since it is **rather mechanical**" (Sefrin-Weis translation, 133).

Proposition 26: Rectification Property of the Quadratrix

Given a square ABCD, and arc BED with center A and radius s (length of a side of the square), if the quadratrix BHT is formed as above, then the ratio of the arc DEB to the line BC equals the ratio of BC to the line CT.

In other words, letting s be the given length of a side of the square (BC), and x be the resulting length CT (where point T is the terminus of the quadratrix),

$\frac{arc\ BED}{BC} = \frac{BC}{CT}$ i.e., $\frac{arc\ BED}{s} = \frac{s}{x}$

[Since arc BED = $\frac{\pi}{2}s$, we have $x = \frac{2}{\pi}s$]

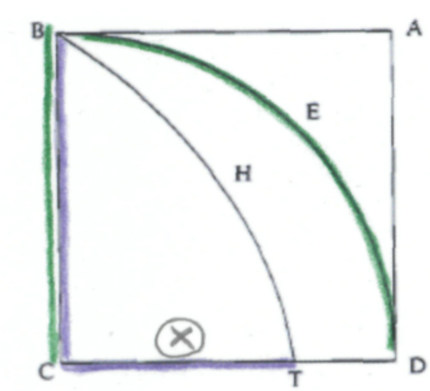


Figure 2: curve BHT is the quadratrix (from Sefrin-Weis)

Proof of Proposition 26 (mainly following Sefrin-Weis):

If the ratio of the arc BED to s is unequal to s/x, then the ratio of arc BED to s must be equal to the ratio of s to a straight line either larger or smaller than x.

Case 1: Assume CK is larger than CT (i.e., larger than x). Line CK describes the arc ZHK with center C (see Figure 3).

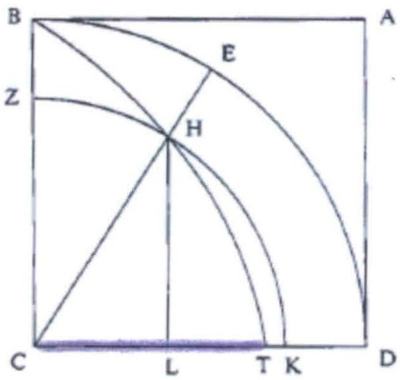


Figure 3: Case 1 (from Sefrin-Weis)

Arc ZHK intersects the quadratrix BHT at point H. The radius CE (of length s) of the larger circle also intersects the quadratrix at point H.

Draw the straight line HL perpendicular to side CD.
Let the length of HL be y.

By assumption,

$$\frac{arc\ BED}{s} = \frac{s}{CK} \qquad (1)$$

Since the ratios of the circumferences of two circles is equal to the ratios of their radii,

$$\frac{arc\ BED}{arc\ ZHK} = \frac{s}{CK} \qquad (2)$$

[Pappus gave explicit proofs of this relation in *Collection* V, 11 and VIII, 22]

Therefore, (1) and (2) imply that arc ZHK has length s:

$$arc\ ZHK = s \qquad (3)$$

By the property (*symptoma*) of the quadratrix,

$$\frac{arc\ BED}{arc\ ED} = \frac{s}{y} \qquad (4)$$

But since arc BED and arc ZHK subtend equal angles, as do arc ED and arc HK,

$$\frac{arc\ BED}{arc\ ED} = \frac{arc\ ZHK}{arc\ HK} \qquad (5)$$

Thus, (4) and (5) imply that

$$\frac{arc\ ZHK}{arc\ HK} = \frac{s}{y} \qquad (6)$$

But equation (3) states that arc ZHK = s.

Therefore, (3) and (6) imply that arc HK = y. However, this is absurd, since any arc of a circle is greater than the chord subtending it [or, by Archimedes *On the Sphere and Cylinder* Assumption 1, of lines having the same end-points the least is a straight line]. Thus, CK is not larger than x.

Case 2: Assume CK is smaller than CT (i.e., smaller than x). Line CK describes the arc ZMK with center C (see Figure 4).

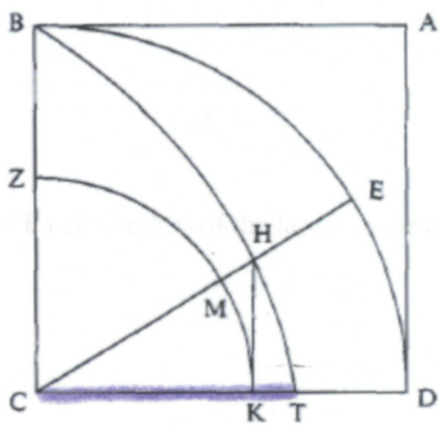


Figure 4: Case 2 (from Sefrin-Weis)

The radius CE (of length s) of the larger circle intersects the quadratrix at point H.

Draw the straight line HK perpendicular to side CD.
Let the length of HK be y.

As in Case I, by assumption,

$$\frac{arc\ BED}{s} = \frac{s}{CK} \qquad (7)$$

and since the ratios of the circumferences of two circles is equal to the ratios of their radii,

$$\frac{arc\ BED}{arc\ ZMK} = \frac{s}{CK} \qquad (8)$$

Thus, (7) and (8) imply that

arc ZMK = s (9)

By the property (*symptoma*) of the quadratrix,

$$\frac{arc\ BED}{arc\ ED} = \frac{s}{y} \qquad (10)$$

But since arc BED and arc ZMK subtend equal angles, as do arc ED and arc MK,

$$\frac{arc\ BED}{arc\ ED} = \frac{arc\ ZMK}{arc\ MK} \qquad (11)$$

Thus, (10) and (11) imply that

$$\frac{arc\ ZMK}{arc\ MK} = \frac{s}{y} \qquad (12)$$

But equation (9) states that arc ZMK = s.

Therefore, (9) and (12) imply that arc MK = y. However, this is absurd, since "any arc of a circle less than a quadrant is less than the portion of the tangent at one extremity of the arc cut off by the radius passing through the other extremity" (Heath, 229) [Archimedes, *On the Sphere and Cylinder* Assumption 2]. Thus, CK is not smaller than x.

Thus, since CK is neither larger than x nor smaller than x, it must equal x.

Proposition 27: Squaring the Circle

It follows from Proposition 26 that for a straight line z such that

$$\frac{z}{s} = \frac{s}{x}$$

then

z = arc BED. Since arc BED is one fourth of the circumference of the circle with radius s,

$$4z = circumference\ of\ circle$$

From this line, it is easy to construct a square equal to a circle itself. For Archimedes has demonstrated that the area of a rectangle with sides equal to the radius and to the circumference of a circle is equal to two times the area of the circle.

[Take side s = 1. Bisect line of measure $\frac{2}{\pi}$ to obtain line of measure $\frac{1}{\pi}$. Construct similar triangles with lines of measure 1 and $\frac{1}{\pi}$ (establishing the proportion $\frac{1}{\pi}$:1 = 1: π) to obtain line of measure π. Then construct the mean proportional between 1 and π to obtain line of measure $\sqrt{\pi}$.]

Propositions 28 and 29 avoid the mechanical generation of the quadratrix and instead apply geometrical analysis by means of loci on surfaces.

Proposition 28: Analytical Determination of the Quadratrix from an Apollonian Helix [cylindrical helix/spiral]

Let the quadrant ABC of a circle be given, and assume that **line BD has been drawn through the interior arbitrarily**, and also that perpendicular to BC is drawn a **line EZ** which **has a given ratio to the arc DC**.

[Note that this differs from Proposition 26, since here we are **given** the ratio of EZ to arc DC, and **thereby given the position of the point E on line BD and of the point Z on line BC**.]

Then **E lies on a uniquely determined line** [i.e., traces a uniquely determined curve in the plane of the quadrant ABC].

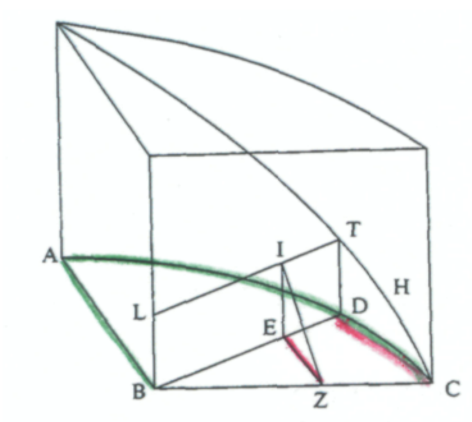


Figure 5 (from Sefrin-Weis)

Proof of Proposition 28 (mainly following Sefrin-Weis):

Construct a cylinder segment over quadrant ABC, and in it take an Apollonian helix as given:

In the surface of the right cylinder over the arc ADC, describe a helix CHT given in position, and let TD be the side of the cylinder.

Draw EI and BL at right angles to the plane of the circle, and draw TL parallel to BD [i.e., create a rectangle BDTL, with **E on BD, I on LT, and EI parallel (and hence equal) to DT**].

Since the ratio of the line EI to the arc DC is **given on account of the helix** [since EI=DT], and since the ratio of EZ to arc DC is given (assumed), then the ratio of EZ to EI will be given also. So ZI, EI are given in position.

Therefore, the joining line ZI is given as well. ZI is also a perpendicular onto BC.
[**That is, the ratios between ZI, EI, EZ, and arc DC will never change, hence also keeping the ascending angle IZE constant, throughout the rotating-cum-ascending motion of line LT**.]

Therefore, **ZI lies in a plane intersecting the cylinder** [although Sefrin-Weis p. 138n observes that there was some historical ambiguity as to whether it is plane EZ/ZI or plane BC/ZI (depicted below by Mendell) which Pappus had in mind; he further asserts that it does not matter] , and **thus I does as well**.

However, **I also lies on a surface belonging to the cylinder**, in particular on the surface created by the helix (for TL travels through both the [rotating] helix THC and the [ascending] straight line LB while it remains parallel throughout to the underlying plane). [That surface is described as a cylindroid or a plectoid (screw-like) by various commentators, ancient and modern.]

Therefore, **I lies on a uniquely determined line** [on "the line of intersection between the aforementioned plane and the surface created by the ascending line BC [sic LT] of the cylindrical helix"].

By projecting that line onto the plane of quadrant ABC, then by construction E lies on a uniquely determined line also.

When the given ratio $\frac{EZ}{arc\ DC} = \frac{AB}{arc\ AC}$, this uniquely determined line will be the quadratrix.

Here are two pictures from Mendell which are helpful in seeing the various surfaces, the lines of intersection, and their projections on the quadrant.

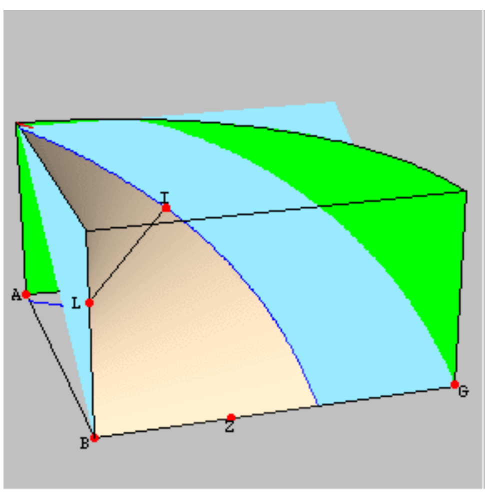


Figure 8 in Mendell

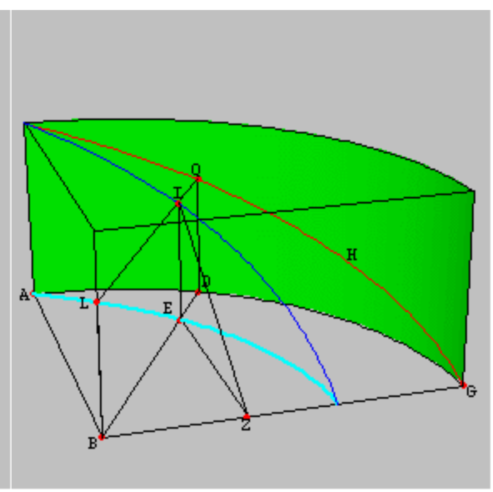
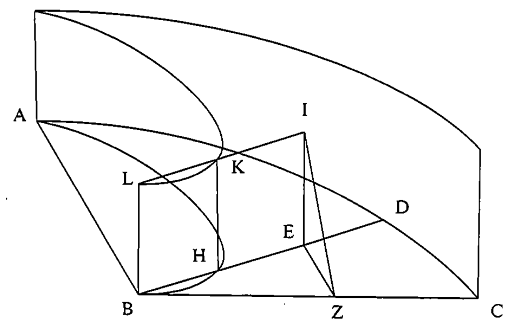


Figure 10 in Mendell

Proposition 29: Analytical Determination of the Quadratrix from the Archimedean Spiral

It [the quadratrix] can also be made subject to analysis by means of a spiral described in the plane, in a similar way [to that done in Proposition 28].



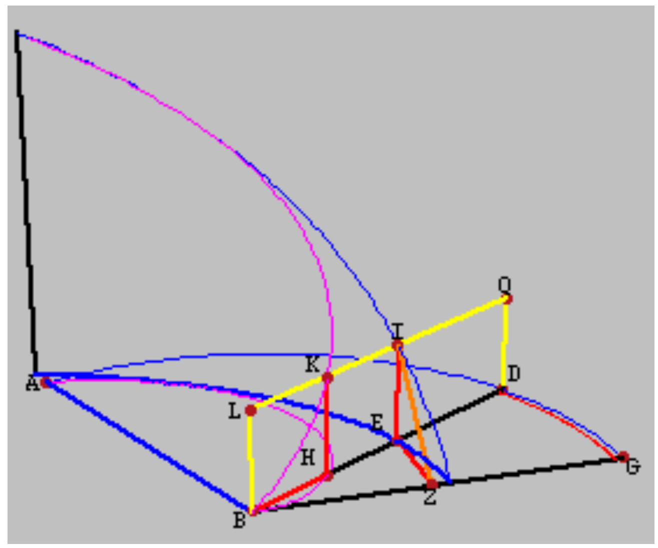


Figure 6 (from Sefrin-Weis) General diagram in Mendell

Proof of Proposition 29:

> [Here I will quote directly from Sefrin-Weis, making parenthetical comments in brackets to explain the meaning and purpose for the ultimate construction and proof.]

For:
Assume that the ratio of EZ to the arc DC is the same as the ratio of AB to the arc ADC [where the arc ADC is a general arc and hence not necessarily the quadrant], and that in the time in which the straight line AB, moving around B passes through the arc ADC, a point on it, starting from A, arrives at B when AB takes the position of CB, and that it creates the spiral BHA.

> [So far, Pappus has both i) generalized the circular arc and ii) constructed on the plane of the arc the spiral BHA by having a point H on AB move along the ray from point B to point A, concurrently as line AB rotates into line CB.]

Then the arc ADC is to the arc CD as AB is to BH, and alternate this equation.

$\frac{arc\ ADC}{arc\ CD} = \frac{AB}{BH}$ and, alternately, $\frac{AB}{arc\ ADC} = \frac{BH}{arc\ DC}$

> [Pappus thus establishes a "generalized" form of the symptoma of the quadratrix – but he applies that generalized form here to the point traced by the spiral BHA, and its distance along the ray AD from the point B.]

But EZ is in that ratio to arc DC also (by assumption). Therefore, BH is equal to ZE. Draw HK at right angles to the plane, equal to BH.

> [Observe that now BH = ZE = HK. Figure 5 in Mendell shows all of the relationships which Pappus has so far established.]

Then K lies in a cylindroid surface over the spiral [depicted in Mendell's Figure 6]

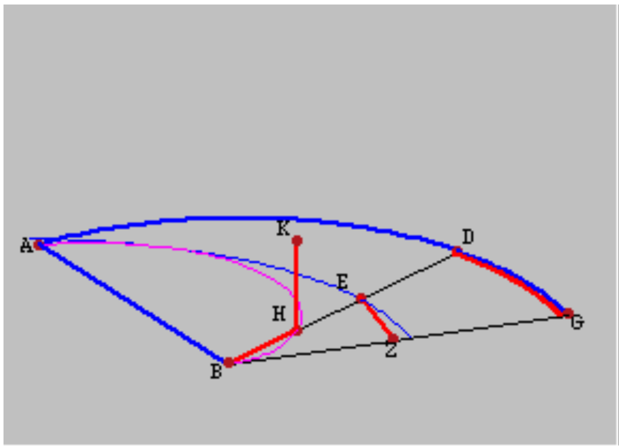


Figure 5 in Mendell

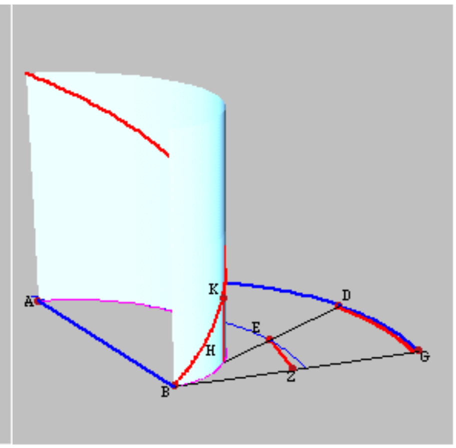


Figure 6 in Mendell

K lies, however, also on the surface of a uniquely determined cone (for BK, when it is joined, turns out to lie on the surface of a cone inclined at an angle of 45 degrees toward the underlying plane, and drawn through the *given* point B as vertex.)

Therefore K lines on a uniquely determined line.

> [Observe that the line is uniquely determined as the intersection of the cylindroid surface over the spiral and the surface of the cone with vertex B. This is depicted in Mendell's Figures 8 and 10).]

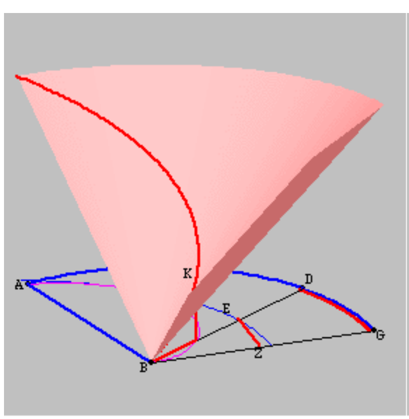


Figure 8 in Mendell

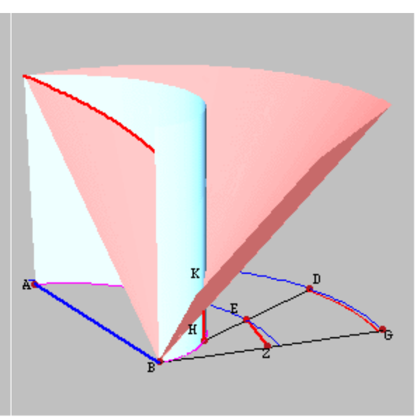


Figure 10 in Mendell

Draw LKI through K as a parallel to EB, and BL and EI at right angles to the underlying plane. Then LKI lies on a plectoid surface (for it travels both through the straight line L, which is *given* in position and through the line, *given* in position, on which K lies.) Therefore, I lies on a uniquely determined surface, also.

> [As in Proposition 28, this surface is created by the rotating-cum-ascending motion of line LKI. That surface is depicted in Mendell's Figure 13.]

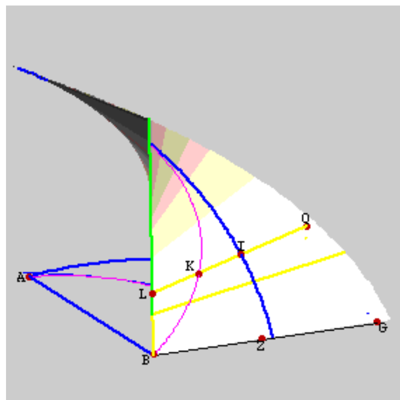


Figure 13 in Mendell

But I also lies on a uniquely determined plane (for ZE is equal to EI, since it is also equal to BH, and ZI turns out to be given as a parallel in position, since it is a perpendicular onto BC).

> [This is depicted in Mendell's Figure 14.]

Therefore, I lies on a uniquely determined line.

> [This line is the intersection of the plane and the plectoid, as depicted in Mendell's Figure 16.]

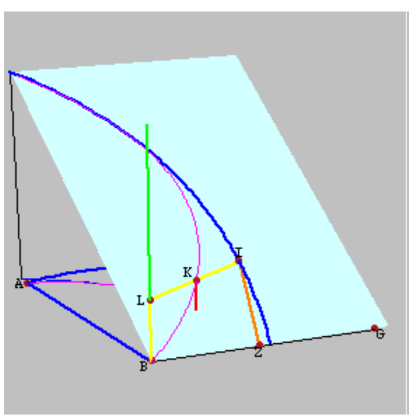


Figure 14 in Mendell

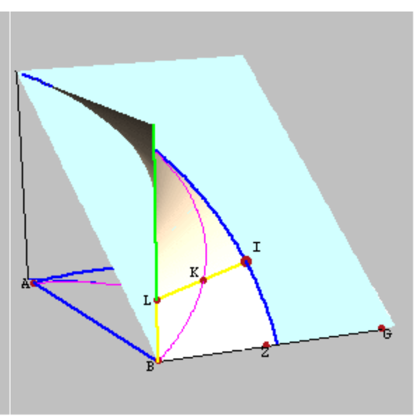


Figure 16 in Mendell

And it is clear that, when the angle ABC is a right angle, the above-mentioned line "quadratrix" comes to be. [That is, when the circular arc is a quadrant.]